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ON THE THEORY OF IGNORABLE DISPLACEMENTS FOR GENERALIZED POINCARÉ–CHETAEV EQUATIONS*

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Ignorable displacements for generalized Poincaré—Chetaev equations /5/ valid for holonomic and nonholonomic systems are defined, and a similar definition is given for the equations of motion of nonholonomic systems in Poincaré—Chetaev variables /6/. It is shown that this definition comprises, as particular cases, the definitions in /1,3,4/, as well as some other definitions of ignorable coordinates in /7—9/ and others. Routh equations of reduced order are presented.

The general theory of ignorable displacements, first introduced in analytic mechanics by Chetaev /1/, was developed for holonomic and nonholonomic systems, respectively, in /2/ and /3,4/.

1. Ignorable displacements and the integral for generalized Poincaré—Chetaev equations. Consider a mechanical system of N material points, defined at every instant of time t by variables $x_1, x_2, \ldots, x_n$ which on real displacements are subjected to $n-l$ linear equations of constraints

$$\eta_j \equiv \sum_{i=1}^{n} a_{ji} x_i^{\cdot} + a_j = 0 \quad (j = l+1, \ldots, n) \tag{1.1}$$

and on possible displacements to $n-l$ relations

$$\omega_j \equiv \sum_{i=1}^{n} a_{ji} \delta x_i = 0 \quad (j = l+1, \ldots, n) \tag{1.2}$$

where l is the number of the system degrees of freedom and a_{ji}, a_j are known functions of variables t and x_i. Nonhomonomic constraints are possible among those considered here. When they are present we assume the last $k-l$ constraints to be holonomic $(l \leqslant k \leqslant n)$.

As shown in /5/, it is then possible to construct a system of displacement operators X_0, $X_1, \ldots, X_l$ that satisfy the relations

$$\begin{gathered}(X_s, X_\alpha) = \sum_{\beta=1}^{l} C_{s\alpha\beta} X_\beta + \sum_{\nu=l+1}^{k} C_{s\alpha\nu} X_\nu \\ (s = 0, 1, \ldots, l; \quad \alpha = 1, \ldots, l)\end{gathered} \tag{1.3}$$

and obtain the equations of motion in the form of generalized Poincaré—Chetaev equations valid for both, the holonomic (when $k = l$) and nonholonomic systems (when $k > l$)

$$\begin{gathered}\frac{d}{dt}\frac{\partial T}{\partial \eta_\alpha} - X_\alpha (T+U) - \sum_{\beta=1}^{l}\Big(C_{0\alpha\beta} + \sum_{s=1}^{l}\eta_s C_{s\alpha\beta}\Big)\frac{\partial T}{\partial \eta_\beta} - \\ \sum_{\nu=l+1}^{k}\Big(C_{0\alpha\nu} + \sum_{s=1}^{l}\eta_s C_{s\alpha\nu}\Big)\Big(\frac{\partial T^\circ}{\partial \eta_\nu}\Big) = 0 \quad (\alpha = 1, \ldots, l)\end{gathered} \tag{1.4}$$

where T is the system kinetic energy (with allowance for imposed constraints), T° is the kinetic energy of the respective "holonomic system" without allowance for nonholonomic constraints, U is the force function, and $C_{s\alpha\beta}$ and $C_{s\alpha\nu}$ are generally functions of variables $t, x_i, X_{l+1}, \ldots, X_k$ which are operators that correspond to the form of the left-hand sides of nonholonomic constraint equations (1.1).

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To obtain some first integrals of Eqs.(1.4) we use the following definition.

Definition 1. We call ignorable any displacement with operator X_γ that satisfies for Eqs.(1.4) the following conditions

1°. Commutators (1.3) for X_γ are of the form

$$(X_s, X_\gamma) = \sum_{\nu=l+1}^{k} C_{s\gamma\nu} X_\nu \quad (s = 0, 1, \ldots, l) \tag{1.5}$$

i.e. they can only be expressed in terms of operators that correspond to right-hand sides of the nonholonomic constraint equations (1.1).

2°. The equality

$$X_\gamma (T + U) + \sum_{\nu=l+1}^{k} \left(C_{0\gamma\nu} + \sum_{s=1}^{l} \eta_s C_{s\gamma\nu}\right) \left(\frac{\partial T^\circ}{\partial \eta_\nu}\right) = 0 \tag{1.6}$$

holds for the kinetic energy and the force function.

Then from the γ-th of Eqs.(1.4) we can obtain the cyclic (ignorable) integral

$$\partial T / \partial \eta_\gamma = \beta_\gamma = \text{const} \tag{1.7}$$

Definition 1 includes, as particular cases, Chetaev's definition of ignorable displacement for holonomic systems, since for them all coefficients $C_{s\gamma\nu}$ are zero, and from (1.5), (1.6) we have Chetaev's conditions and the definition given in /4/ for the same Eqs.(1.4), since when operator X_γ is communtative with all operators X_s, conditions (1.5), (1.6) are the same as those in /4/.

2. Ignorable displacements for equations of motion of nonholonomic systems in Poincaré—Chetaev variables. As in the case of equations of motion of nonholonomic systems in Poincaré—Chetaev's variables /6/

$$\frac{d}{dt}\frac{\partial T}{\partial \eta_\alpha} - Y_\alpha (T + U) - \sum_{\beta=1}^{l} \left(k_{0\alpha\beta} + \sum_{s=1}^{l} \eta_s k_{s\alpha\beta}\right) \frac{\partial T}{\partial \eta_\beta} - \sum_{\nu=l+1}^{k} \left(k'_{0\alpha\nu} + \sum_{s=1}^{l} \eta_s k'_{s\alpha\nu}\right) \left(\frac{\partial T^\circ}{\partial \eta_\nu}\right) = 0 \quad (\alpha = 1, 2, \ldots, l) \tag{2.1}$$

it is possible to formulate the following definition.

Definition 2. Operator Y_γ is an operator of ignorable displacement for Eqs.(2.1), when the followings conditions are satisfied:

$$(Y_s, Y_\gamma) = \sum_{\nu=l+1}^{k} k'_{s\gamma\nu} X_\nu \quad (s = 0, 1, \ldots, l) \tag{2.2}$$

$$Y_\gamma (T + U) + \sum_{\nu=l+1}^{k} \left(k'_{0\gamma\nu} + \sum_{s=1}^{l} \eta_s k'_{s\gamma\nu}\right) \left(\frac{\partial T^\circ}{\partial \eta_\nu}\right) = 0$$

where Y_α are operators of real displacements of nonhomonomic systems with constraints

$$\eta_\nu = \sum_{\alpha=1}^{l} c_{\nu\alpha} \eta_\alpha + c_{\nu 0} \quad (\nu = l + 1, \ldots, k) \tag{2.3}$$

For the respective holonomic system these operators are expressed in terms of displacements operators X_α in the form

$$Y_\alpha = X_\alpha + \sum_{\nu=l+1}^{k} c_{\nu\alpha} X_\nu \quad (\alpha = 0, 1, \ldots, l) \tag{2.4}$$

$$(Y_s, Y_\alpha) = \sum_{\beta=1}^{l} k_{s\alpha\beta} Y_\beta + \sum_{\nu=l+1}^{k} k'_{s\alpha\nu} X_\nu$$

$$(s = 0, 1, \ldots, l; \quad \alpha = 1, \ldots, l)$$

$$k_{s\alpha\beta} = C_{s\alpha\beta} + \sum_{\nu=l+1}^{k} c_{\nu\alpha} C_{s\nu\beta} + \sum_{\mu=l+1}^{k} c_{\mu s} \left(C_{\mu\alpha\beta} + \sum_{\nu=l+1}^{k} c_{\nu\alpha} C_{\mu\nu\beta}\right)$$

$$k'_{s\alpha\nu} = k_{s\alpha\nu} - \sum_{\beta=1}^{l} c_{\nu\beta} k_{s\alpha\beta} + Y_s (c_{\nu\alpha}) - Y_\alpha (c_{\nu s})$$

$$(s = 0, 1, \ldots, l; \ \alpha = 1, \ldots, l; \ \beta = 1, \ldots, k; \ \nu = l + 1, \ldots, k)$$

where the coefficients $C_{s\alpha\beta}$ are taken from the cummutators

$$(X_s, X_\alpha) = \sum_{\beta=1}^{k} C_{s\alpha\beta} X_\beta \quad (s = 0, 1, \ldots, k; \quad \alpha = 1, \ldots, k)$$

T° is the kinetic energy of the respective holonomic system, and T its expression with (2.3) taken into account, i.e. it is the kinetic energy of the nonholonomic system.

To each ignorable displacement corresponds in conformity with definition 2 the ignorable integral

$$\partial T / \partial \eta_\gamma = \beta_\gamma = \text{const} \tag{2.5}$$

Definition 2 comprises, as a particular case, the definition given in /3/ for the equations of motion of nonholonomic systems in Poincaré—Chetaev variables with constraint multipliers, according to which X_γ is an ignorable displacement when

$$\begin{aligned} &(X_s, X_\gamma) = 0, \quad (s = 0, 1, \ldots, k), \; X_\gamma (L^\circ) = 0 \\ &c_{\nu\gamma} = 0 \; (\nu = l + 1, \ldots, k) \end{aligned} \tag{2.6}$$

where X_α are displacement operators of the respective holonomic system, $c_{\nu\alpha}$ are coefficients in the nonholonomic constraint equations (2.3), and L° is the Lagrange function for the corresponding holonomic system.

Indeed, the substitution of (2.6) into (2.4) yields

$$Y_\alpha = X_\alpha + \sum_{\nu=l+1}^{k} c_{\nu\alpha} X_\nu, \quad Y_\gamma = X_\gamma \quad (\alpha = 0, 1, \ldots, l; \quad \alpha \neq \gamma)$$

$$(Y_s, Y_\gamma) = (Y_s, X_\gamma) = -\sum_{\nu=l+1}^{k} X_\gamma (c_{\nu s}) X_\nu \quad (s = 0, 1, \ldots, l)$$

$$Y_\gamma (L) = X_\gamma (L) = \sum_{\nu=l+1}^{k} \frac{\partial T^\circ}{\partial \eta_\nu} \left[X_\gamma (c_{\nu 0}) + \sum_{s=1}^{l} \eta_s X_\gamma (c_{\nu s}) \right]$$

The last two formulas show that the conditions are satisfied, since $k'_{s\gamma\nu} = -X_\gamma (c_{\nu s})$, $k'_{0\gamma\nu} = -X_\gamma (c_{\nu 0})$. Hence when condition (2.6) is satisfied, X_γ is also an ignorable displacement of Eqs.(2.1) (transformed to independent parameters), while existence of the ignorable integral $\partial T^\circ / \partial \eta_\gamma = \text{const}$ for equations with multipliers implies the existence of an ignorable integral for the transformed equations (2.1). The latter can be obtained directly, since

$$\frac{\partial T}{\partial \eta_\gamma} = \frac{\partial T^\circ}{\partial \eta_\gamma} + \sum_{\nu=l+1}^{k} \frac{\partial T^\circ}{\partial \eta_\nu} c_{\nu\gamma} \tag{2.7}$$

Using definition (2.2) it is also possible to prove the following.

a) The definition of ignorable coordinate in /7/ which yields the first integral for equations with multipliers does not, generally, provide the integral for the transformed equations (2.1). To obtain the latter it is necessary and sufficient that the respective second condition (2.2) is satisfied.

b) The definition of ignorable coordinate in /8/ reduces to the particular case of definition (2.2), it yields first integrals simultaneously for equations with multipliers and for the transformed equations (2.1).

Indeed, let the nonholonomic system in generalized coordinates be subjected to nonholonomic constraints

$$q_\nu^{\cdot} = \sum_{\alpha=1}^{l} c_{\nu\alpha} q_\alpha^{\cdot} \quad (\nu = l + 1, \ldots, n) \tag{2.8}$$

and the equations with multipliers be

$$\begin{aligned} &\frac{d}{dt} \frac{\partial L^\circ}{\partial q_\alpha^{\cdot}} - \frac{\partial L^\circ}{\partial q_\alpha} - \sum_{\nu=l+1}^{n} \mu_\nu c_{\nu\alpha} = 0 \\ &\frac{d}{dt} \frac{\partial L^\circ}{\partial q_\nu^{\cdot}} - \frac{\partial L^\circ}{\partial q_\nu} + \mu_\nu = 0 \quad (\alpha = 1, \ldots, l; \; \nu = l + 1, \ldots, n) \end{aligned} \tag{2.9}$$

Then the ignorable coordinate that satisfies conditions /7/

$$\frac{\partial L^\circ}{\partial q_\gamma} = 0, \quad \sum_{\nu=l+1}^{n} \mu_\nu c_{\nu\gamma} = 0 \tag{2.10}$$

yields the first integral $\partial T^\circ/\partial q_\gamma^{\cdot} = \text{const}$. But formula (2.7) shows that without the assumption that $c_{\gamma\gamma} = 0$, the integral $\partial T/\partial q_\gamma^{\cdot} = \text{const}$ for the transformed equations, which in this case are Voronets' equations, does not always follow from (2.10).

By virtue of (2.10) it is generally possible to obtain from (2.7)

$$\frac{d}{dt}\frac{\partial T}{\partial q_\gamma^{\cdot}} = \sum_{\nu=l+1}^{n} \frac{dc_{\nu\gamma}}{dt}\frac{\partial T^\circ}{\partial q_\nu^{\cdot}} + \sum_{\nu=l+1}^{n} c_{\nu\gamma}\frac{d}{dt}\frac{\partial T^\circ}{\partial q_\nu^{\cdot}}$$

or

$$\sum_{\nu=l+1}^{n} c_{\nu\gamma}\frac{d}{dt}\frac{\partial T^c}{\partial q_\nu^{\cdot}} = \sum_{\nu=l+1}^{n} c_{\nu\gamma}\frac{d}{dt}\frac{\partial L^\circ}{\partial q_\nu^{\cdot}} = \frac{d}{dt}\frac{\partial T}{\partial q_\gamma^{\cdot}} - \sum_{\nu=l+1}^{n} \frac{\partial T^\circ}{\partial q_\nu^{\cdot}} \sum_{s=1}^{l} q_s^{\cdot} Y_s(c_{\nu\gamma}) \tag{2.11}$$

where from (2.3)

$$Y_0 = \frac{\partial}{\partial t}, \quad Y_s = \frac{\partial}{\partial q_s} + \sum_{\nu=l+1}^{n} c_{\nu s}\frac{\partial}{\partial q_\nu}$$

$$(Y_s, Y_\gamma) = \sum_{\nu=l+1}^{n} k'_{s\gamma\nu} X_\nu, \quad X_\nu = \frac{\partial}{\partial q_\nu}$$

$$k'_{0\gamma\nu} = 0, \quad k'_{s\gamma\nu} = Y_s(c_{\nu\gamma}) - Y_\gamma(c_{\nu s}) \quad (s = 1, \ldots, l;\ \nu = l+1, \ldots, n)$$

moreover

$$Y_\gamma(T+U) = \frac{\partial L^\circ}{\partial q_\gamma} + \sum_{\nu=l+1}^{n} c_{\nu\gamma}\frac{\partial L^c}{\partial q_\nu} + \sum_{\nu=l+1}^{n} \frac{\partial T^\circ}{\partial q_\nu^{\cdot}} \sum_{s=1}^{l} q_s^{\cdot} Y_\gamma(c_{\nu s})$$

which by virtue of the first of conditions (2.10) yields

$$\sum_{\nu=l+1}^{n} c_{\nu\gamma}\frac{\partial L^\circ}{\partial q_\nu} = Y_\gamma(T+U) - \sum_{\nu=l+1}^{n} \frac{\partial T^\circ}{\partial q_\nu^{\cdot}} \sum_{s=1}^{l} q_s^{\cdot} Y_\gamma(c_{\nu s}) \tag{2.12}$$

Substituting (2.11) and (2.12) into the second of conditions (2.10), we obtain

$$\sum_{\nu=l+1}^{n} \mu_\nu c_{\nu\gamma} = \frac{d}{dt}\frac{\partial T}{\partial q_\gamma^{\cdot}} - Y_\gamma(T+U) - \sum_{\nu=l+1}^{n} \frac{\partial T^\circ}{\partial q_\nu^{\cdot}} \sum_{s=1}^{l} q_s^{\cdot}[Y_s(c_{\nu\gamma}) - Y_\gamma(c_{\nu s})] = 0 \tag{2.13}$$

Formula (2.13) proves the second part of statement a).

If the ignorable coordinate q_γ satisfies for Eqs.(2.9) the conditions /8/

$$\partial L^\circ/\partial q_\gamma = 0, \quad c_{\nu\gamma} = 0 \quad (\nu = l+1, \ldots, n)$$

then it satisfies conditions (2.10), and it follows from (2.7) that $\partial T/\partial q_\gamma^{\cdot} = \text{const}$ is the integral of transformed equations (2.1), and from (2.3) and (2.13) that conditions (2.2) are also satisfied. Statement b) is proved.

Another definition was given in /9–11/, namely that q_γ is an ignorable coordinate, if

$$\frac{\partial L^\circ}{\partial q_\gamma} = 0, \quad \frac{\partial c_{\nu\alpha}}{\partial q_\gamma} = 0 \quad (\alpha = 1, \ldots, l;\quad \nu = l+1, \ldots, n)$$

As shown in /12/, this definition does not provide the first integral of equations with multipliers. Neither can it yield an integral for the transformed equations (2.1), i.e. for Voronets equations, since the corresponding equation for q_γ is then of the form

$$\frac{d}{dt}\frac{\partial T}{\partial q_\gamma^{\cdot}} = \sum_{\nu=l+1}^{n} c_{\nu\gamma}\frac{\partial(T+U)}{\partial q_\nu} + \sum_{\nu=l+1}^{n} \frac{\partial T^c}{\partial q_\nu^{\cdot}} \sum_{s=1}^{l} q_s^{\cdot} \times \left(\frac{\partial c_{\nu\gamma}}{\partial q_s} + \sum_{\mu=l+1}^{n} c_{\mu s}\frac{\partial c_{\nu\gamma}}{\partial q_\mu} - \sum_{\mu=l+1}^{n} c_{\mu\gamma}\frac{\partial c_{\nu s}}{\partial q_\mu}\right) \tag{2.14}$$

which shows that it is possible to obtain the integral $\partial T/\partial q_\gamma^c = \text{const}$ only when one more condition that follows from (2.2), according to which the right-hand side of (2.14) must vanish, is satisfied.

The definition of an ignorable coordinate given in /13/ for Voronets' equations is in the form of conditions

$$\partial T/\partial q_\gamma = 0, \quad c_{\nu\gamma} = 0$$
$$\partial c_{\nu s}/\partial q_\gamma = 0 \quad (s = 1, \ldots, l;\ \nu = l+1, \ldots, n)$$

It can be shown that these conditions reduce to conditions (2.2).

Thus definition 2 comprizes, as particular cases, definitions given in /3,8,13/, while for the definitions of such coordinates in /7,9,11/ to yield the first integral for the transformed equations (2.1) (which in this case are the Voronets' equations) it is necessary to satisfy some additional conditions derived from (2.2).

3. The Routh equations. Let us show that having $l-m$ ignorable integrals (1.7)

$$\partial T/\partial \eta_\gamma = \beta_\gamma = \text{const} \quad (\gamma = m+1, \ldots, l) \tag{3.1}$$

it is possible to reduce the order of the system of Eqs.(1.4). Indeed, since T is a positive definite quadratic function of parameters $\eta_1, \ldots, \eta_l$, it is possible to solve (3.1) for, let us say, the last $l-m$ parameters

$$\eta_\gamma = \eta_\gamma(t, x_1, \ldots, x_n, \eta_1, \ldots, \eta_m, \beta_{m+1}, \ldots, \beta_l) \quad (\gamma = m+1, \ldots, l) \tag{3.2}$$

Introducing, now, the Routh function of the form /1/

$$R = T + U - \sum_{\gamma=m+1}^{l} \eta_\gamma \beta_\gamma \tag{3.3}$$

by virtue of (3.1) we have

$$\delta R = \sum_{\alpha=1}^{l} \omega_\alpha X_\alpha(T+U) + \sum_{\alpha=1}^{m} \frac{\partial T}{\partial \eta_\alpha}\delta\eta_\alpha - \sum_{\gamma=m+1}^{l} \eta_\gamma \delta\beta_\gamma \tag{3.4}$$

On the other hand, by substituting (3.2) into (3.3) we obtain function R dependent on t; x_i; $\eta_1, \ldots, \eta_m$; $\beta_{m+1}, \ldots, \beta_l$, and

$$\delta R = \sum_{\alpha=1}^{l} \omega_\alpha X_\alpha(R) + \sum_{\alpha=1}^{m} \frac{\partial R}{\partial \eta_\alpha}\delta\eta_\alpha + \sum_{\gamma=m+1}^{l} \frac{\partial R}{\partial \beta_\gamma}\delta\beta_\gamma \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{gathered}\partial T/\partial\eta_\alpha = \partial R/\partial\eta_\alpha,\ X_\alpha(T+U) = X_\alpha(R) \\ \eta_\gamma = -\partial R/\partial\beta_\gamma,\quad X_\gamma(T+U) = X_\gamma(R) \quad (\alpha = 1, \ldots, m; \\ \gamma = m+1, \ldots, l)\end{gathered} \tag{3.6}$$

By virtue of (3.6) the first m generalized Poincaré—Chetaev equations (1.4) assume the form

$$\begin{gathered}\frac{d}{dt}\frac{\partial R}{\partial \eta_\alpha} - X_\alpha(R) - \sum_{\beta=1}^{m}\Big(C_{0\alpha\beta} + \sum_{s=1}^{m}\eta_s C_{s\alpha\beta}\Big)\frac{\partial R}{\partial\eta_\beta} - \sum_{\gamma=m+1}^{l}\beta_\gamma\Big(C_{0\alpha\gamma} + \sum_{s=1}^{m}\eta_s C_{s\alpha\gamma}\Big) - \\ \sum_{\nu=l+1}^{k}\Big(C_{0\alpha\nu} + \sum_{s=1}^{m}\eta_s C_{s\alpha\nu} - \sum_{\gamma=m+1}^{l}\frac{\partial R}{\partial\beta_\gamma}C_{\gamma\alpha\nu}\Big)\Big(\frac{\partial T^*}{\partial\eta_\nu}\Big) = 0, \quad (\alpha = 1, \ldots, m)\end{gathered} \tag{3.7}$$

which are the Routh equations of order $2m\ (m < l)$ which together with the kinetic equations

$$\frac{dx_i}{dt} = X_0(x_i) + \sum_{s=1}^{m}\eta_s X_s(x_i) - \sum_{\gamma=m+1}^{l}\frac{\partial R}{\partial\beta_\gamma}X_\gamma(x_i) \quad (i = 1, \ldots, n) \tag{3.8}$$

constitute a closed system of $n+m$ differential equations for the determination of $n+m$ variables x_i and η_α as functions of time t. After this it is possible to determine η_γ using Eqs.(3.6).

In the case of holonomic systems Eqs.(3.7) coincide with the Routh equations for the Poincaré equations /1/, and when the operators of ignorable variables X_α are commutative with all other operators, they assume the form given in /4/.

Similary, when Eqs.(2.1) have $l-m$ ignorable integrals (3.1), the corresponding Routh equations of the reduced system are

$$\frac{d}{dt}\frac{\partial R}{\partial \eta_\alpha} - Y_\alpha(R) - \sum_{\beta=1}^{m}\left(k_{0\alpha\beta} + \sum_{s=1}^{m}\eta_s k_{s\alpha\beta}\right)\frac{\partial R}{\partial \eta_\beta} - \sum_{\gamma=m+1}^{l}\beta_\gamma\left(k_{0\alpha\gamma} + \sum_{s=1}^{m}\eta_s k_{s\alpha\gamma}\right) - \tag{3.9}$$
$$\sum_{\nu=l+1}^{k}\left(k'_{0\alpha\nu} + \sum_{s=1}^{m}\eta_s k'_{s\alpha\nu} - \sum_{\gamma=m+1}^{l}\frac{\partial R}{\partial \beta_\gamma}k'_{\gamma\alpha\nu}\right)\left(\frac{\partial T^\circ}{\partial \eta_\nu}\right) = 0 \qquad (\alpha = 1, \ldots, m)$$

where the Routh function is defined by the relation (3.3).

4. Examples. 1°. A mechanical system consists of a table rotating about its vertical axis with moment of inertia J on which moves a Chaplygin sledge /14/. We define the system position by the following variables: the table angle of rotation ψ, rectangular coordinates ξ, η of the sledge (cutter) of its contact point A with the table, and the sledge angle of turn φ about point A. Between these variables there is a single nonholonomic constraint

$$\eta_4 = \xi^\cdot \sin\varphi - \eta^\cdot \cos\varphi = 0 \tag{4.1}$$

Taking these variables as the Poincaré—Chetaev variables and the quantities

$$\eta_1 = \varphi^\cdot, \quad \eta_2 = \xi^\cdot\cos\varphi + \eta^\cdot\sin\varphi, \quad \eta_3 = \psi^\cdot \tag{4.2}$$

as the parameters of real displacements, we obtain the following system of nonholonomic displacement operators:

$$X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial \varphi}, \quad X_2 = \cos\varphi\frac{\partial}{\partial \xi} + \sin\varphi\frac{\partial}{\partial \eta}, \quad X_3 = \frac{\partial}{\partial \psi}$$

The commutators of these operators are zero, except the following one:

$$(X_1, X_2) = -X_4; \quad X_4 = \sin\varphi\partial/\partial\xi - \cos\varphi\partial/\partial\eta$$

where X_4 is the operator which corresponds to the left-hand side of the nonholonomic constraint equation (4.1). These operators satisfy the first of conditions (2.2). In addition we have

$$\begin{aligned} &T = {}^1/_2 m(\gamma^2\eta_1^2 + \eta_2^2 - 2b\eta_1\eta_2 + 2\Delta_1\eta_1\eta_3 - 2\Delta_2\eta_2\eta_3 + \Delta_3\eta_3^2) \\ &T^\circ = {}^1/_2 m(\eta_4^2 - 2a\eta_1\eta_4 - \Delta_4\eta_4\eta_3 + \ldots, \; U = 0 \\ &\Delta_1 = \gamma^2 + a(\xi\cos\varphi + \eta\sin\varphi) + b(-\xi\sin\varphi + \eta\cos\varphi) \\ &\Delta_2 = b + \xi\sin\varphi - \eta\cos\varphi, \quad \Delta_3 = J/m + \gamma^2 + \xi^2 + \eta^2 + \\ &\quad 2a(\xi\cos\varphi + \eta\sin\varphi) + 2b(-\xi\sin\varphi + \eta\cos\varphi), \quad \Delta_4 = a + \\ &\quad \xi\cos\varphi + \eta\sin\varphi \end{aligned} \tag{4.3}$$

where m is the mass of the sledge with a, b its center of mass in the system of coordinates Axy rigidly attached to it (axis Ax directed along the /sledge/ blade, and the Ay axis is normal to it) and the ellipsis denotes terms that are independent of η_4. It will be seen that only X_3 satisfies the second of conditions (2.2). The integral

$$\partial T/\partial\eta_3 = m(\Delta_1\eta_1 - \Delta_2\eta_2 + \Delta_3\eta_3) = \beta = \text{const} \tag{4.4}$$

that corresponds to this ignorable displacement, is the integral of the system moment of momentum about the vertical axis of rotation which can be directly obtained using the general theorem of dynamics.

For the derivation of Routh's equations we have from (4.4) the expression for η_3, and obtain the Routh function $R = T + U - \beta\eta_3$. These equations are of the form

$$\left(\gamma^2 - \frac{\Delta_1^2}{\Delta_3}\right)\eta_1^\cdot - \left[b - (2b - \Delta_2)\frac{\Delta_1}{\Delta_3}\right]\eta_2^\cdot - \frac{\Delta_1}{\Delta_3}\frac{\partial\Delta_1}{\partial\varphi}\eta_1^2 + \left(a - \frac{\Delta_1\Delta_4}{\Delta_3}\right)\eta_1\eta_3 - \frac{1}{\Delta_3^2}(\beta/m - \Delta_1\eta_1 + \Delta_2\eta_2) \times \tag{4.5}$$
$$\left[(\beta/m + \Delta_1\eta_1 + \Delta_2\eta_2)\frac{\partial\Delta_1}{\partial\varphi} - 2(a\Delta_3 - \Delta_1\Delta_4)\eta_2\right] = 0, \quad \left[b - (2b - \Delta_2)\frac{\Delta_1}{\Delta_3}\right]\eta_1^\cdot - \left[1 - \frac{(2b - \Delta_2)^2}{\Delta_3}\right]\eta_2^\cdot +$$
$$\left(a - \frac{2b - \Delta_2}{\Delta_3}\frac{\partial\Delta_1}{\partial\varphi}\right)\eta_1^2 - (2b - \Delta_2)\frac{\Delta_4}{\Delta_3}\eta_1\eta_2 - \frac{1}{\Delta_3^2}[\beta/m - \Delta_1\eta_1 + (2b - \Delta_2)\eta_2]\left\{2\left[a\Delta_3 - (2b - \Delta_2)\frac{\partial\Delta_1}{\partial\varphi}\right]\eta_1 - \right.$$
$$\left.[\beta/m - \Delta_1\eta_1 + 3(2b - \Delta_2)\eta_2]\Delta_4\right\} = 0$$

which with equations

$$\varphi^\cdot = \eta_1, \quad \xi^\cdot = \eta_2\cos\varphi, \quad \eta^\cdot = \eta_2\sin\varphi$$

constitute a closed system for the determination of $\varphi, \xi, \eta, \eta_1, \eta_2$ as functions of t. After this we can determine ψ using Eqs.(4.4), in which $\psi^{\cdot} = \eta_3$.

2°. A small wheel with a sharp rim, part of some instrument which pushes it on a horizontal plane, while continuously holding the wheel rim in a vertical plane. The wheel rolls without friction on the horizontal plane with the wheel center of mass on the vertical axis passing through the contact point /15,16/.

We define the position of this system by the horizontal coordinates ξ, η of the wheel center of mass (of the contact point) G, by the angle ψ between the projection of the wheel plane on the horizontal plane and the axis $O\xi$, and by the angle of turn φ of the wheel about its axis O_1O_2. Between them there are nonholonomic constraints which stipulate that the wheel rolls without friction and that the velocity of its center of mass remains all the time in its plane:

$$\eta_3 \equiv \xi^{\cdot} - R\varphi^{\cdot} \sin\varphi = 0, \quad \eta_4 \equiv \eta^{\cdot} - R\varphi^{\cdot} \cos\psi = 0 \tag{4.6}$$

We assume ξ, η, φ, ψ to be Poincaré—Chetaev variables, and $\eta_1 = \varphi^{\cdot}$, $\eta_2 = \psi^{\cdot}$ to be the real displacement parameters. The respective displacement operators are then

$$X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial\varphi} + R\sin\psi\frac{\partial}{\partial\xi} + R\cos\psi\frac{\partial}{\partial\eta}, \quad X_2 = \frac{\partial}{\partial\psi}$$

These operators satisfy condition (1.5), since

$$(X_0, X_1) = 0, \quad (X_0, X_2) = 0, \quad (X_1, X_2) = -R\cos\psi X_3 + R\sin\psi X_4$$

where $X_3 = \partial/\partial\xi$, $X_4 = \partial/\partial\eta$ are operators which correspond to the left-hand of Eqs.(4.6) of nonholonomic constraints. We have, moreover,

$$T = {}^1/_2 m\,[(R^2 + k_1^2)\,\eta_1^2 + k_2^2\eta_2^2], \quad U = 0$$
$$T^{\circ} = {}^1/_2 m\,[\eta_3^2 + \eta_4^2 + 2R\eta_1\,(\eta_3\sin\psi + \eta_4\cos\psi) + \ldots]$$

where k_1, k_2 are the radii of inertia of the wheel about its diameter and its axis of rotation, /respectively/, and the ellipsis denotes terms free of η_3, η_4. Hence condition (1.6) is satisfied for X_1 and X_2, and the respective ignorable integrals are

$$\partial T/\partial\eta_1 = m\,(R^2 + k_1^2)\,\eta_1 = \beta_1 = \text{const}, \; \partial T/\partial\eta_2 = mk_2^2\eta_2 = \beta_2 = \text{const}$$

These integrals imply that parameters η_1, η_2 remains always constant $\eta_1 = \beta_1/[m(R^2 + k_1^2)]$, $\eta_2 = \beta_2/(mk_2^2)$. Their substitution into Eqs.(3.9)

$$\varphi^{\cdot} = \eta_1, \; \psi^{\cdot} = \eta_2, \; \xi^{\cdot} = R\sin\psi\eta_1, \; \eta^{\cdot} = R\cos\psi\eta_2$$

followed by integration, yields the sought laws of motion of the wheel.

3°. A homogeneous sphere of radius a rolls without slipping on a fixed horizontal plane. As in /17/, we define the position of the sphere by the coordinates ξ, η, ζ of its center of mass and by Euler's angles θ, φ, ψ, and take them as Poincaré—Chetaev variables. The condition of rolling without slipping on the plane provides the equations of constraints

$$\eta_4 = \xi^{\cdot} - a\,(\theta^{\cdot}\sin\psi - \varphi^{\cdot}\cos\psi\sin\theta) = 0$$
$$\eta_5 \equiv \eta^{\cdot} + a\,(\theta^{\cdot}\cos\psi + \varphi^{\cdot}\sin\psi\sin\theta) = 0, \quad \eta_6 = \zeta^{\cdot} = 0$$

Taking $\eta_1 = \theta^{\cdot}, \eta_2 = \varphi^{\cdot}, \eta_3 = \psi^{\cdot} + \varphi\cos\theta$ as the parameters of real displacements, we obtain the system of displacement operators

$$X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial\theta} + a\sin\psi\frac{\partial}{\partial\xi} - a\cos\psi\frac{\partial}{\partial\eta}$$
$$X_2 = \frac{\partial}{\partial\varphi} - \cos\theta\frac{\partial}{\partial\psi} - a\cos\psi\sin\theta\frac{\partial}{\partial\xi} - a\sin\psi\sin\theta\frac{\partial}{\partial\eta}, \quad X_3 = \frac{\partial}{\partial\psi}$$

of which only X_3 satisfies condition (1.5), since

$$(X_0, X_1) = 0, \; (X_0, X_2) = 0, \; (X_0, X_3) = 0$$
$$(X_1, X_2) = \sin\theta X_3 - a\cos\theta\cos\psi X_4 - a\cos\theta\sin\psi X_5$$
$$(X_2, X_3) = -a\sin\psi\sin\theta X_4 + a\cos\psi\sin\theta X_5$$
$$(X_3, X_1) = a\cos\psi X_4 + a\sin\psi X_5$$

where $X_4 = \partial/\partial\xi$, $X_5 = \partial/\partial\eta$ are operators that correspond to the left-hand side of the equations of nonholonomic constraints $\eta_4 = 0$, $\eta_5 = 0$.

Moreover,

$$T = {}^1/_2[(A + ma^2)\,(\eta_1^2 + \eta_2^2) + A\eta_3^2], \quad U = 0$$
$$T^0 = {}^1/_2 m[\eta_4^2 + \eta_5^2 + \eta_6^2 + 2(a\sin\psi\eta_1 - a\cos\psi\sin\theta)\,\eta_4 + 2\,(-a\cos\psi\eta_1 - a\sin\psi\sin\theta\eta_2)\,\eta_5 + \ldots].$$

Hence X_3 satisfies also condition (1.6). From this we have the ignorable integral

$$\partial T/\partial \eta_3 = A\eta_3 = \beta = \text{const}$$

To derive Routh's equations we introduce the function

$$R = T + U - \beta\eta_3 = {}^1/_2 \{(A + ma^2)(\eta_1{}^2 + \eta_2{}^2) - \beta^2/A\}$$

which after substitution into (3.8) yields

$$(A + ma^2)\eta_1{}^{\cdot} + ma^2 \sin\theta\cos\theta\eta_2{}^2 + \frac{A + ma^2}{A}\beta\sin\theta\eta_2 = 0$$

$$(A + ma^2)\eta_2{}^{\cdot} - ma^2 \sin\theta\cos\theta\eta_1\eta_2 - \frac{A + ma^2}{A}\beta\sin\theta\eta_1 = 0$$

These equations together with

$$\theta^{\cdot} = \eta_1,\ \varphi^{\cdot} = \eta_2,\ \psi^{\cdot} = -\eta_1\cos\theta + \beta/A$$
$$\xi^{\cdot} = a(\eta_1 \sin\psi - \eta_2\cos\psi\sin\theta)$$
$$\eta^{\cdot} = -a(\eta_1\cos\psi + \eta_2\sin\psi\sin\theta),\ \zeta^{\cdot} = 0$$

derived from (3.9) constitute a closed system for the determination of the unknown quantities $\theta, \varphi, \psi, \xi, \eta, \zeta, \eta_1, \eta_2$ as functions of time t.

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